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A sufficient condition for two long-range orders coexisting in a lattice many-body system

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Abstract. In this paper, we prove a sufficient condition for two long-range orders being either present or absent simultaneously in the absolute ground state of a lattice many-body boson or fermion model. As an application of this theorem, we give a simplified proof on the coexistence of the resonating valence bond (RVB) long-range order and the on-site-pairing long-range order in the ground state of the Hubbard model.

In the study of many-body systems, the existence of various long-range correlations in their ground states is a problem of fundamental importance. In general, the presence of a long-range order in the ground state of the system will completely determine the low-temperature behaviour of the system and leads to some of the most interesting phenomena, such as magnetism, superfluidity and superconductivity. However, as far as the analytical work is concerned, the existence of a specific long-range order in a concrete model is notoriously difficult to be established on a rigorous basis. For example, the famous Heisenberg model was proposed in 1928 [1] to explain the magnetic properties of insulators. However, it took about 40 years to show that, in one or two dimensions, the magnetic long-range orders do not exist in this model, when the temperature $T \neq 0$ [2]. For the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg model, the existence of an antiferromagnetic long-range order in three dimensions at a non-zero temperature was established much later [3] and the existence of this long-range order in two dimensions, $T = 0$, still remains an open problem [4]. Here, a great challenge to theorists is to find the proper methods for studying a specified model.

In a previous paper [5], we introduced a new method to show the absence of some long-range orders in a lattice many-body system. More precisely, we proved the following theorem.

Theorem 1. Let $H_\Lambda = H_0 + V$ be the Hamiltonian of a lattice many-body system, where H_0 represents the kinetic energy of particles and V is the interaction Hamiltonian. Assume that V is short-ranged and its intensity $|V|$ is bounded. Let $\hat{A}_i$ and $\hat{B}_i$ be two localized operators centered at site i. If they satisfy the following commutation relation

$$[H_\Lambda, \hat{A}_i] = \alpha \hat{B}_i \tag{1}$$

where $\alpha \neq 0$ is a complex constant, then the absolute ground state Ψ_0 of H_Λ does not have a long-range order of operator $\hat{B}_i$.

As an application of theorem 1, in [6], we proved that, in the *doped* negative-U Hubbard model, the so-called η-pairing long-range order [7, 8] is actually absent. In [9], we further applied theorem 1 to study the effect of a staggered crystalline field on the antiferromagnetic long-range order in some antiferromagnetic models.

In this paper, we shall further pursue this commutator approach and prove a sufficient condition for two long-range orders coexisting in a lattice many-body system. Then, as an application of this new theorem, we show that, when the Hubbard model is doped, the on-site pairing long-range order must coexist with the so-called resonating valence bond (RVB) order [10, 11] in the absolute ground states of the Hubbard Hamiltonian.

Before stating our theorem in a precise form, we would like to recall several definitions and introduce some useful notation.

In solid-state physics, most of the interesting models are defined on a lattice. Consequently, their Hamiltonians have a discrete form. Take a finite lattice Λ with N_Λ lattice sites and let H_Λ be the Hamiltonian of a many-body boson or fermion model defined on Λ. Then, the Hilbert space of H_Λ is of the following form:

$$V_\Lambda = \prod_{i\in\Lambda} \otimes V_i \tag{2}$$

where V_i is the relevant configuration space at site i (for instance, for the Hubbard model, V_i is spanned by $|0\rangle$, the empty-site configuration, $|\uparrow\rangle$ and $|\downarrow\rangle$, the singly-occupied configurations, as well as $|\uparrow\downarrow\rangle$, the doubly-occupied configuration). In terms of V_Λ, H_Λ can now be written as a matrix. Let $\hat{N}$ be the total number operator of particles in the system and μ be the chemical potential. We denote the absolute ground state of $H_\Lambda - \mu\hat{N}$ by $\Psi_0(\mu, \Lambda)$.

Let $\hat{G}_i$ be a localized operator defined at site i. Following Yang [12], we define a reduced density matrix $\mathcal{M}(\hat{G}_i) = (M_{ij})$ by

$$M_{ij} \equiv \langle\Psi_0(\mu, \Lambda)|\hat{G}_i^\dagger \hat{G}_j|\Psi_0(\mu, \Lambda)\rangle. \tag{3}$$

Then, $\Psi_0(\mu, \Lambda)$ has a long-range order of $\hat{G}_i$ if and only if the largest eigenvalue $\lambda_{\max}$ of $\mathcal{M}(\hat{G}_i)$ satisfies the condition

$$\lambda_{\max} \geqslant DN_\Lambda \tag{4}$$

where $D > 0$ is a constant independent of N_Λ, as $N_\Lambda \to \infty$ and $N/N_\Lambda \to \rho \neq 0$. In fact, in [12], Yang explicitly showed that, if inequality (4) holds, then

$$\lim_{|i-j|\to\infty} \lim_{N_\Lambda\to\infty} \langle\Psi_0(\mu, \Lambda)|\hat{G}_i^\dagger \hat{G}_j|\Psi_0(\mu, \Lambda)\rangle \neq 0 \tag{5}$$

must also hold.

By applying the variational principle, condition (4) for $\Psi_0(\mu, \Lambda)$ having a long-range order of $\hat{G}_i$ can be slightly relaxed. Let $f(i)$ be a complex function defined on lattice Λ. $f(i)$ is called admissible if $|f(i)| = 1$. By the variational principle, we have

$$\lambda_{\max} \geqslant \sum_{i,j\in\Lambda} \overline{f(i)} M_{ij} f(j) \equiv \langle f|\mathcal{M}(\hat{G}_i)|f\rangle \tag{6}$$

holds for any admissible function $f(i)$. Therefore, if $\langle f_0|\mathcal{M}(\hat{G}_i)|f_0\rangle \geqslant DN_\Lambda$ holds for some admissible function $f_0(i)$, inequality (4) must be true. In particular, when Λ is a finite d-dimensional hypercubic lattice with the periodic boundary condition, we may choose an admissible function as $f_q(i) = (1/\sqrt{N_\Lambda})\exp(-\mathrm{i}q\cdot i)$. Consequently,

$$\begin{aligned}\langle f_q|\mathcal{M}(\hat{G}_i)|f_q\rangle &= \frac{1}{N_\Lambda}\sum_{i,j\in\Lambda} \langle\Psi_0(\mu, \Lambda)|\hat{G}_i^\dagger \hat{G}_j|\Psi_0(\mu, \Lambda)\rangle \exp(\mathrm{i}q\cdot(i-j)) \\ &\equiv \langle\Psi_0(\mu, \Lambda)|\hat{G}^\dagger(q)\hat{G}(q)|\Psi_0(\mu, \Lambda)\rangle\end{aligned} \tag{7}$$

where $\hat{G}_{\boldsymbol{q}} \equiv (1/\sqrt{N_\Lambda}) \sum_{i\in\Lambda} \hat{G}_i \exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{i})$ and $\boldsymbol{q} = (q_1, \ldots, q_d)$ is a reciprocal vector of Λ. In this case, the presence of a long-range ordering of $\hat{G}_i$ in the ground state $\Psi_0(\mu, \Lambda)$ may be thought as a Bose–Einstein condensate of the $\hat{G}$-wave at some reciprocal vector $\boldsymbol{q}_0$, which is characterized by

$$\langle\Psi_0(\mu, \Lambda)|\hat{G}^\dagger_{\boldsymbol{q}_0}\hat{G}_{\boldsymbol{q}_0}|\Psi_0(\mu, \Lambda)\rangle \geqslant DN_\Lambda. \tag{8}$$

With these definitions and notation, we now summarize our new results in the following theorem.

Theorem 2. Let $H_\Lambda = H_0 + V$ be the Hamiltonian of a lattice many-body model. We assume that the interaction V is short-ranged with a characteristic radius R and its intensity $|V| \leqslant U_0 < \infty$. Let $\hat{A}_i$, $\hat{B}_i$ and $\hat{C}_i$ be some localized operators defined on lattice Λ. If these operators satisfy the following commutation relation

$$[H_\Lambda - \mu\hat{N}, \hat{A}_i] = \beta\hat{B}_i + \gamma\hat{C}_i \tag{9}$$

where $\beta \neq 0$ and $\gamma \neq 0$ are some complex constants, then the long-range orders of $\hat{B}_i$ and $\hat{C}_i$ must be either present or absent simultaneously in the absolute ground state $\Psi_0(\mu, \Lambda)$ of the Hamiltonian $H_\Lambda - \mu\hat{N}$.

Before proceeding to the proof of this theorem, let us see why these statements should be true by a less rigorous but plausible argument. Ignoring mathematical rigour temporarily, we consider the Hamiltonian H_∞ defined on the *whole and infinite* lattice and let Ψ_0 be an absolute ground state of H_∞. (Indeed, one should be more careful when one handles these objects. For instance, the expectation value E_0 of H_∞ in Ψ_0 is actually $-\infty$ and, hence, is ill defined.) Now, let us formally calculate the expectation value of commutator (9) in Ψ_0. We obtain

$$\begin{aligned}\langle\Psi_0|[H_\infty - \mu\hat{N}, \hat{A}_i]|\Psi_0\rangle &= (E_0 - E_0)\langle\Psi_0|\hat{A}_i|\Psi_0\rangle = 0\\ &= \beta\langle\Psi_0|\hat{B}_i|\Psi_0\rangle + \gamma\langle\Psi_0|\hat{C}_i|\Psi_0\rangle.\end{aligned} \tag{10}$$

Therefore, as $\beta \neq 0$ and $\gamma \neq 0$, we should have

$$\langle\Psi_0|\hat{B}_i|\Psi_0\rangle = -\frac{\gamma}{\beta}\langle\Psi_0|\hat{C}_i|\Psi_0\rangle. \tag{11}$$

In physics, if a many-body system has a long-range order of $\hat{B}_i$, one will expect that the expectation value of $\hat{B}_i$ in Ψ_0 does not vanish. Consequently, by identity (11), the expectation value of $\hat{C}_i$ in Ψ_0 must also be non-zero. Therefore, the system has a long-range order of $\hat{C}_i$, too. In other words, the two long-range orders are simultaneously present in the system.

In the following, we shall establish theorem 2 on a mathematically rigorous basis by studying the well-defined correlation functions of $\hat{A}_i$, $\hat{B}_i$ and $\hat{C}_i$ on a *finite lattice* Λ. Then, in the last step of our proof, we shall take the thermodynamic limit.

Proof of the theorem. First, we notice that, in terms of the correlation functions of $\hat{B}_i$ and $\hat{C}_i$, the theorem can be re-formulated in the following equivalent form. Under the condition of theorem 2, as $N_\Lambda \to \infty$ and $N/N_\Lambda \to \rho \neq 0$, inequality

$$\langle\Psi_0(\mu, \Lambda)|\hat{B}^\dagger(f)\hat{B}(f)|\Psi_0(\mu, \Lambda)\rangle \geqslant D_1 N_\Lambda \tag{12}$$

holds for some admissible function $f(\boldsymbol{i})$, if and only if

$$\langle\Psi_0(\mu,\Lambda)|\hat{C}^\dagger(f)\hat{C}(f)|\Psi_0(\mu,\Lambda)\rangle \geqslant D_2 N_\Lambda \tag{13}$$

also holds for the same function $f(\boldsymbol{i})$. In equations (12) and (13), $\hat{G}(f) \equiv (1/\sqrt{N_\Lambda})\sum_{i\in\Lambda} f(\boldsymbol{i})\hat{G}_{\boldsymbol{i}}$, where $\hat{G}_{\boldsymbol{i}} = \hat{B}_{\boldsymbol{i}}$ or $\hat{C}_{\boldsymbol{i}}$. Naturally, the constants $D_1 > 0$ and $D_2 > 0$ may not be equal.

Now, we take an arbitrary admissible function $f(\boldsymbol{i})$ and rewrite commutator (9) as

$$[H_\Lambda - \mu\hat{N}, \hat{A}(f)] = \beta\hat{B}(f) + \gamma\hat{C}(f) \equiv \hat{K}(f). \tag{14}$$

Consider the correlation function of $\hat{K}(f)$. Apparently, we have

$$\begin{aligned} 0 &\leqslant \langle\Psi_0(\mu,\Lambda)|\hat{K}^\dagger(f)\hat{K}(f)|\Psi_0(\mu,\Lambda)\rangle \\ &\leqslant \langle\Psi_0(\mu,\Lambda)|\hat{K}^\dagger(f)\hat{K}(f)|\Psi_0(\mu,\Lambda)\rangle + \langle\Psi_0(\mu,\Lambda)|\hat{K}(f)\hat{K}^\dagger(f)|\Psi_0(\mu,\Lambda)\rangle \\ &\equiv S_{\hat{K}}(f). \end{aligned} \tag{15}$$

Introducing a complete set of the eigenvectors $\{\Psi_n(\mu,\Lambda)\}$ of $H_\Lambda - \mu\hat{N}$, $S_{\hat{K}}(f)$ can be further written as

$$\begin{aligned} S_{\hat{K}}(f) &= \sum_n (|\langle\Psi_n(\mu,\Lambda)|\hat{K}(f)|\Psi_0(\mu,\Lambda)\rangle|^2 + |\langle\Psi_n(\mu,\Lambda)|\hat{K}^\dagger(f)|\Psi_0(\mu,\Lambda)\rangle|^2) \\ &= \sum_n \left\{ \frac{|\langle\Psi_n(\mu,\Lambda)|\hat{K}(f)|\Psi_0(\mu,\Lambda)\rangle|}{\sqrt{E_n - E_0}} (|\langle\Psi_n|\hat{K}(f)|\Psi_0\rangle|\sqrt{E_n - E_0}) \right. \\ &\quad \left. + \frac{|\langle\Psi_n(\mu,\Lambda)|\hat{K}^\dagger(f)|\Psi_0(\mu,\Lambda)\rangle|}{\sqrt{E_n - E_0}} (|\langle\Psi_n|\hat{K}^\dagger(f)|\Psi_0\rangle|\sqrt{E_n - E_0}) \right\}. \end{aligned} \tag{16}$$

In the last line of equation (16), we used the fact that $\Psi_0(\mu,\Lambda)$ is the absolute ground state of H_Λ and, hence, $\sqrt{E_n - E_0}$ is well defined. However, there is one dangerous point which should be clarified here. Assume that some eigenvector Ψ_n is degenerate with Ψ_0. In this case, we have $\sqrt{E_n - E_0} = 0$. That will make the corresponding fraction term in equation (16) ill defined. Fortunately, this danger can be easily eliminated by applying commutator (14) to the numerator of the term. In fact, by the commutation relation, we have

$$\langle\Psi_n(\mu,\Lambda)|\hat{K}(f)|\Psi_0(\mu,\Lambda)\rangle = (E_n - E_0)\langle\Psi_n(\mu,\Lambda)|\hat{A}(f)|\Psi_0(\mu,\Lambda)\rangle = 0 \tag{17}$$

when $E_n - E_0 = 0$. Similarly, we can also show that $\langle\Psi_n|\hat{K}^\dagger|\Psi_0\rangle = 0$. Consequently, the fraction term can be simply put equal to zero.

Next, we apply the Cauchy–Schwarz inequality $(\sum_n a_n b_n) \leqslant \sqrt{(\sum_n |a_n|^2)}\sqrt{(\sum_n |b_n|^2)}$ to the right-hand side of equation (16). We obtain

$$\begin{aligned} S_{\hat{K}}(f) &\leqslant \sqrt{\sum_n (|\langle\Psi_n|\hat{K}(f)|\Psi_0\rangle|^2 + |\langle\Psi_n|\hat{K}^\dagger(f)|\Psi_0\rangle|^2)(E_n - E_0)} \\ &\quad \times \sqrt{\sum_n \frac{|\langle\Psi_n|\hat{K}(f)|\Psi_0\rangle|^2 + |\langle\Psi_n|\hat{K}^\dagger(f)|\Psi_0\rangle|^2}{E_n - E_0}}. \end{aligned} \tag{18}$$

The first factor on the right-hand side of inequality (18) is simply equal to

$$m(\hat{K}(f)) \equiv \sqrt{\langle\Psi_0(\mu,\Lambda)|[\hat{K}^\dagger(f), [H_\Lambda - \mu\hat{N}, \hat{K}(f)]]|\Psi_0(\mu,\Lambda)\rangle}. \tag{19}$$

On the other hand, by using commutation relation (14), the second factor on the right-hand side of inequality (18) can be simplified as

$$\sqrt{\sum_n \frac{|\langle\Psi_n|\hat{K}^\dagger(f)|\Psi_0\rangle|^2 + |\langle\Psi_n|\hat{K}(f)|\Psi_0\rangle|^2}{E_n - E_0}}$$
$$= \sqrt{\sum_n (|\langle\Psi_n|\hat{A}^\dagger(f)|\Psi_0\rangle|^2 + |\langle\Psi_n|\hat{A}(f)|\Psi_0\rangle|^2)(E_n - E_0)}$$
$$= \sqrt{\langle\Psi_0(\mu,\Lambda)|[\hat{A}^\dagger(f),[H_\Lambda - \mu\hat{N},\hat{A}(f)]]|\Psi_0(\mu,\Lambda)\rangle}$$
$$\equiv m(\hat{A}(f)). \tag{20}$$

Therefore, we have

$$0 \leqslant \langle\Psi_0(\mu,\Lambda)|\hat{K}^\dagger(f)\hat{K}(f)|\Psi_0(\mu,\Lambda)\rangle \leqslant S_{\hat{K}}(f) \leqslant m(\hat{E}(f))m(\hat{A}(f)). \tag{21}$$

An important observation is that, under the condition of theorem 2, both $m(\hat{K}(f))$ and $m(\hat{A}(f))$ are quantities of $O(1)$ as N_Λ tends to infinity. Therefore, the correlation function of $\hat{K}_{\boldsymbol{i}}$ at 'momentum f' is, at most, a quantity of $O(1)$. For definiteness, let us consider $m(\hat{A}(f))$. By its definition, $m^2(\hat{A}(f))$ is the expectation value of the commutator $[A^\dagger(f),[H_\Lambda - \mu\hat{N},\hat{A}(f)]]$ in the absolute ground state $\Psi_0(\Lambda)$. Since $\hat{A}_{\boldsymbol{i}}$ is a localized operator and V is short-ranged, the commutator must be of the following form,

$$[A^\dagger(f),[H_\Lambda - \mu\hat{N},\hat{A}(f)]] = \frac{1}{N_\Lambda}\sum_{\boldsymbol{i},\boldsymbol{j}\in\Lambda}[A_{\boldsymbol{i}}^\dagger,[H_\Lambda - \mu\hat{N},\hat{A}_{\boldsymbol{j}}]]\overline{f(\boldsymbol{i})}f(\boldsymbol{j}) = \frac{1}{N_\Lambda}\sum_{\boldsymbol{i}\in\Lambda}\hat{O}_{\boldsymbol{i}}(f) \tag{22}$$

where $\{\hat{O}_{\boldsymbol{i}}(f)\}$ are some localized operators dependent of $f(\boldsymbol{i})$ (if V is not short-ranged, then $\{\hat{O}_{\boldsymbol{i}}(f)\}$ may be not localized). In general, they are polynomials of the creation and annihilation operators of bosons or fermions. These statements can be easily checked by calculating the commutator for a concrete model, say the Hubbard model. Therefore, we have

$$0 \leqslant m^2(\hat{A}(f)) \leqslant \frac{1}{N_\Lambda} \times N_\Lambda \max_{\boldsymbol{i},f} |\langle\Psi_0(\Lambda)|\hat{O}_{\boldsymbol{i}}(f)|\Psi_0(\Lambda)\rangle|. \tag{23}$$

On the other hand, since $|V|$ is bounded, we can find a constant $L > 0$, which is independent of $\boldsymbol{i}$, f and Λ, such that

$$|\langle\Psi_0(\Lambda)|\hat{O}_{\boldsymbol{i}}(f)|\Psi_0(\Lambda)\rangle| \leqslant L \tag{24}$$

holds for all the localized operators $\{\hat{O}_{\boldsymbol{i}}(f)\}$. Combining equations (22)–(24), we obtain

$$m^2(\hat{A}(f)) = O(1) \tag{25}$$

as $N_\Lambda \to \infty$.

Remark 1. In fact, inequality (24) can be formally proven by using the well known Gershgorin theorem in matrix theory [13]. This theorem tells us that, for an $n \times n$ matrix $\underline{A} = (a_{ij})$, each of its eigenvalues satisfies the following inequality:

$$|\lambda| \leqslant \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ij}|. \tag{26}$$

We apply this theorem to estimate the expectation value $\langle\Psi_0(\Lambda)|\hat{O}_i(f)|\Psi_0(\Lambda)\rangle$. Let $\lambda_{\max}$ and $\lambda_{\min}$ be the maximal and minimal eigenvalues of the operator $\hat{O}_i(f)$, respectively. By the variational principle and the Gershgorin theorem, we should have

$$|\langle\Psi_0(\Lambda)|\hat{O}_i(f)|\Psi_0(\Lambda)\rangle| \leqslant \max(|\lambda_{\min}|, |\lambda_{\max}|) \leqslant \max_m \sum_j |[\underline{O_i(f)}]_{mj}| \tag{27}$$

where $\underline{O_i(f)}$ is the matrix representation of the operator $\hat{O}_i(f)$ in terms of a specific basis. On the other hand, since $\hat{O}_i(f)$ is a localized operator, the number of non-vanishing terms in each row of $\underline{O_i(f)}$ cannot exceed a constant proportional to R, the range of the interaction potential V. Furthermore, each non-zero matrix element has an absolute value bounded by a positive constant M_0, which is roughly proportional to $|V|$. Consequently, inequality (24) holds if we choose $L = O(R|V|)$.

We expand the correlation function of $\hat{K}_i$ as

$$\begin{aligned}\langle\Psi_0(\mu,\Lambda)|\hat{K}^\dagger(f)\hat{K}(f)|\Psi_0(\mu,\Lambda)\rangle \\ &= |\beta|^2\langle\Psi_0(\Lambda)|\hat{B}^\dagger(f)\hat{B}(f)|\Psi_0(\Lambda)\rangle + \overline{\beta}\gamma\langle\Psi_0(\Lambda)|\hat{B}^\dagger(f)\hat{C}(f)|\Psi_0(\Lambda)\rangle \\ &\quad +\beta\overline{\gamma}\langle\Psi_0(\Lambda)|\hat{C}^\dagger(f)\hat{B}(f)|\Psi_0(\Lambda)\rangle + |\gamma|^2\langle\Psi_0(\Lambda)|\hat{C}^\dagger(f)\hat{C}(f)|\Psi_0(\Lambda)\rangle.\end{aligned} \tag{28}$$

By shifting the mixing terms to the right-hand side of inequality (21) and applying the Cauchy–Schwarz inequality to these terms, we can further write the inequality as

$$\begin{aligned}|\beta|^2\langle\Psi_0(\Lambda)|\hat{B}^\dagger(f)\hat{B}(f)|\Psi_0(\Lambda)\rangle + |\gamma|^2\langle\Psi_0(\Lambda)|\hat{C}^\dagger(f)\hat{C}(f)|\Psi_0(\Lambda)\rangle \\ &\leqslant m(\hat{A}(f))m(\hat{K}(f)) - \overline{\beta}\gamma\langle\Psi_0|\hat{B}^\dagger(f)\hat{C}(f)|\Psi_0\rangle - \overline{\gamma}\beta\langle\Psi_0|\hat{C}^\dagger(f)\hat{B}(f)|\Psi_0\rangle \\ &\leqslant m(\hat{A}(f))m(\hat{K}(f)) + 2|\beta||\gamma|\sqrt{\langle\Psi_0|\hat{B}^\dagger(f)\hat{B}(f)|\Psi_0\rangle\langle\Psi_0|\hat{C}^\dagger(f)\hat{C}(f)|\Psi_0\rangle}.\end{aligned} \tag{29}$$

This inequality implies theorem 2.

First, let us assume that $\Psi_0(\mu,\Lambda)$ has a long-range order of $\hat{B}_i$ at momentum f but does not support a long-range order of $\hat{C}_i$. We shall show that this assumption is in contradiction with inequality (29). By the definition of long-range orders, there should be a constant $D > 0$, which is independent of N_Λ, such that

$$\langle\Psi_0(\mu,\Lambda)|\hat{B}^\dagger(f)\hat{B}(f)|\Psi_0(\mu,\Lambda)\rangle \geqslant DN_\Lambda \tag{30}$$

as $N_\Lambda \to \infty$. Consequently, the left-hand side of inequality (29) is a quantity of order $O(N_\Lambda)$. On the other hand, since $\Psi_0(\mu,\Lambda)$ has no long-range orders of $\hat{C}_i$, the correlation function of $\hat{C}_i$ is, at most, a quantity of $O(1)$ in the thermodynamic limit. Therefore, the right-hand side of inequality (29) can be, at most, a quantity of order $O(\sqrt{N_\Lambda})$ since the product $m(\hat{A}(f))m(\hat{K}(f))$ is of order $O(1)$ as we showed above. Consequently, inequality (29) will be eventually violated as $N_\Lambda \to \infty$. Therefore, $\Psi_0(\mu,\Lambda)$ must also have a long-range order of $\hat{C}_i$ at the same momentum f.

Similarly, we can show that, if $\Psi_0(\mu,\Lambda)$ has a long-range order of $\hat{C}_i$ at momentum f, it must also have a long-range order of $\hat{B}_i$ at the same momentum. Otherwise, inequality (29) is violated in the thermodynamic limit. Therefore, the long-range orders of $\hat{B}_i$ and $\hat{C}_i$ must be either present or absent simultaneously in the absolute ground state $\Psi_0(\mu,\Lambda)$ of the Hamiltonian $H_\Lambda - \mu\hat{N}$.

Our proof is accomplished. □

Remark 2. Gershgorin's theorem is a very powerful tool in finding bounds to the eigenvalues of a specific matrix. Therefore, it has been widely used in studying many-body models defined on lattices. For instance, in [14], we applied it to simplify the proof of the Nagaoka theorem [15].

As an application of theorem 2, let us consider the *doped* Hubbard model [16]. On lattice Λ, the Hubbard Hamiltonian is of the following form,

$$H_\Lambda = -t \sum_\sigma \sum_{\langle ij \rangle} (c^\dagger_{i\sigma} c_{j\sigma} + c^\dagger_{j\sigma} c_{i\sigma}) + U \sum_{i \in \Lambda} n_{i\uparrow} n_{i\downarrow} \tag{31}$$

where $c^\dagger_{i\sigma}$ ($c_{i\sigma}$) is the fermion creation (annihilation) operator which creates (annihilates) a fermion with spin σ at lattice site $\boldsymbol{i}$. $n_{i\uparrow} = c^\dagger_{i\uparrow} c_{i\uparrow}$. $\langle \boldsymbol{ij} \rangle$ denotes a pair of nearest-neighbour sites of the lattice. $t > 0$ and U are two parameters representing the kinetic energy and on-site interaction between fermions, respectively. Here, we shall allow U to be either positive or negative. It is easy to see that this Hamiltonian commutes with the total fermion number operator $\hat{N}$. Consequently, the number of fermions is a good quantum number and the Hilbert space of H_Λ can be divided into numerous subspaces $\{V(N)\}$. Each of them is characterized by an integer N and is called a sector. In particular, when $N = N_\Lambda$, the number of lattice sites, the corresponding subspace is called half-filled. Other sectors are called doped with either holes ($N < N_\Lambda$) or electrons ($N > N_\Lambda$).

Originally, this model was introduced to explain the Mott metal–insulator transitions [17]. After the discovery of high-temperature superconductivity in the rare-earth-based copper oxides, Anderson and his collaborators [10, 11] proposed that the physical properties of these materials can be described by a two-dimensional Hubbard model and the ground state of this model should be an RVB state. By definition, an RVB state $\tilde{\Psi}$ is characterized by a non-vanishing expectation value of the following operator

$$\hat{b}_{\langle ij \rangle} \equiv c_{i\uparrow} c_{j\downarrow} - c_{i\downarrow} c_{j\uparrow} \tag{32}$$

within it. Here, $\boldsymbol{i}$ and $\boldsymbol{j}$ are a pair of nearest-neighbour lattice sites. We would like to emphasize that $\tilde{\Psi}$ is an eigenstate of H_∞, not H_Λ. For any eigenstate Ψ_Λ of H_Λ, the expectation value of $\hat{b}_{\langle ij \rangle}$, which contains only particle annihilation operators, will be identically zero because Ψ_Λ is also an eigenvector of the total particle number operator $\hat{N}$. On the other hand, $\tilde{\Psi}$ may not be an eigenvector of $\hat{N}$ if spontaneous symmetry breaking occurs. Based on this proposal, Anderson developed a new theory for high-temperature superconductivity.

However, as research went into depth, more and more results showed evidence which disfavoured the existence of the RVB states in the Hubbard model. In particular, in a paper of 1990 [18], Zhang noticed that the Hubbard Hamiltonian H_Λ satisfies the following commutation relation:

$$[H_\Lambda - \mu \hat{N}, c_{i\uparrow} c_{i\downarrow}] = t \sum_j \hat{b}_{\langle ij \rangle} - (1 - 2\mu) U c_{i\uparrow} c_{i\downarrow}. \tag{33}$$

Therefore, by using the less rigorous argument mentioned at the beginning of this paper, Zhang showed that, when $\mu \neq \frac{1}{2}$ (it corresponds to the doped cases), if the absolute ground state of the model is an RVB state, it must also have an on-site pairing long-range order. However, that is impossible for a *positive-U* Hubbard model. In [5] and [19], we made Zhang's argument rigorous by studying the correlation functions of the operators $\hat{b}_{\langle ij \rangle}$ and $c_{i\uparrow} c_{i\downarrow}$ in the absolute ground state $\Psi_0(\mu, \Lambda)$ of the Hubbard Hamiltonian $H_\Lambda - \mu \hat{N}$. We proved the following theorem.

Theorem 3. When the Hubbard model is doped, the RVB long-range order and the on-site-pairing long-range order must be either present or absent simultaneously in its absolute ground state.

However, our proof had some short-comings. First, the proof was heavily dependent on the fact that, in this special case, operator $c_{i\uparrow}c_{i\downarrow}$ appears on both sides of commutation relation (33). Therefore, it is very difficult to generalize the proof to other cases, in which operators $\hat{A}_I$, $\hat{B}_i$ and $\hat{C}_i$ are distinct. Second, this previous proof was complicated and inelegant. With the newly-proved theorem 2 in this paper, these problems can be simultaneously solved.

Taking $\hat{A}_i = \hat{C}_i = c_{i\uparrow}c_{i\downarrow}$ and $\hat{B}_i = \sum_j \hat{b}_{\langle ij\rangle}$, we see that commutation relation (9) is satisfied with $\beta = t$ and $\gamma = -(1-2\mu)U$. Therefore, by theorem 2, the on-site-pairing long-range order of $c_{i\uparrow}c_{i\downarrow}$ and the long-range order of $\sum_j \hat{b}_{\langle ij\rangle}$ must be either present or absent simultaneously in the doped Hubbard model.

In literature, $\sum_j \hat{b}_{\langle ij\rangle}$ is called the s-wave RVB operator. Therefore, a more accurate form of theorem 3 is the following corollary of theorem 2.

Corollary. When the Hubbard model is doped, an on-site-pairing long-range order must co-exist with an s-wave RVB long-range order in the absolute ground state of the Hamiltonian.

On the physical basis, one expects that an on-site-pairing long-range order exists in the absolute ground state of a doped *negative-U* Hubbard model, which is a phenomenological model widely used to study the superconductivity in a lattice fermion model. Consequently, as the corollary of theorem 2 tells us, an RVB long-range order should also exist in the doped *negative-U* Hubbard model rather than in the *positive-U* Hubbard model.

In summary, in this paper, we prove a sufficient condition for two long-range orders being either present or absent simultaneously in the absolute ground state of a lattice many-body model. As an application of this theorem, we give a simplified proof of a previous theorem on the coexistence of the RVB long-range order and the on-site-pairing long-range order in the absolute ground state of the doped Hubbard model.

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